Maximal classes for some families of Darboux-like and quasicontinuous-like functions

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Abstract. The paper is a survey concerning maximal additive, multiplicative and latticelike classes for certain families of functions similar to quasicontinuous or Darboux functions.

Keywords: maximal addditive class, maximal multiplicative class, quasicontinuous functions, Darboux functions.

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1. Introduction

Let X be a topological space and let $\mathcal{F}$ be a nonempty family of real functions defined on X. For $\mathcal{F}$, we define the maximal additive class $\mathcal{M}_{add}(\mathcal{F})$ as

$$\mathcal{M}_{add}(\mathcal{F}) = \{f : X \to \mathbb{R}; f + g \in \mathcal{F} \text{ for every } g \in \mathcal{F}\},$$

the maximal multiplicative class $\mathcal{M}_{mult}(\mathcal{F})$ as

$$\mathcal{M}_{mult}(\mathcal{F}) = \{f : X \to \mathbb{R}; f \cdot g \in \mathcal{F} \text{ for every } g \in \mathcal{F}\},$$

the maximal class with respect tu maximum $\mathcal{M}_{max}(\mathcal{F})$ as

$$\mathcal{M}_{max}(\mathcal{F}) = \{f : X \to \mathbb{R}; \max(f, g) \in \mathcal{F} \text{ for every } g \in \mathcal{F}\},$$

the maximal class with respect to minimum $\mathcal{M}_{min}(\mathcal{F})$ as

$$\mathcal{M}_{min}(\mathcal{F}) = \{f : X \to \mathbb{R}; \min(f, g) \in \mathcal{F} \text{ for every } g \in \mathcal{F}\},$$

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and the maximal latticelike class $\mathcal{M}_{latt}(\mathcal{F})$ as

$$\mathcal{M}_{latt}(\mathcal{F}) = \{f : X \to \mathbb{R}; \max(f,g) \in \mathcal{F} \text{ and } \min(f,g) \in \mathcal{F} \text{ for every } g \in \mathcal{F}\}.$$

The notion of maximal classes for certain family of functions might be used for the first time in the Bruckner's monograph [8]. The maximal additive class is always nonempty because the zero constant function 0 (i.e. $f(x) = 0$ for all $x \in X$) belongs to $\mathcal{M}_{add}(\mathcal{F})$ for each family $\mathcal{F}$. Similarly, the maximal multiplicative family is always nonempty because the constant function 1 ($f(x) = 1$) belongs to $\mathcal{M}_{mult}(\mathcal{F})$. The classes $\mathcal{M}_{max}(\mathcal{F})$, $\mathcal{M}_{min}(\mathcal{F})$ and $\mathcal{M}_{latt}(\mathcal{F})$ can be empty (e.g for quasicontinuous functions with closed graph). Further, if the constant function 0 belongs to $\mathcal{F}$ then $\mathcal{M}_{add}(\mathcal{F}) \subset \mathcal{F}$, and if the constant function 1 belongs to $\mathcal{F}$ then $\mathcal{M}_{mult}(\mathcal{F}) \subset \mathcal{F}$. If $\mathcal{F}$ is closed under addition then $\mathcal{F} \subset \mathcal{M}_{add}(\mathcal{F})$, and if $\mathcal{F}$ is closed under multiplication then $\mathcal{F} \subset \mathcal{M}_{mult}(\mathcal{F})$. So, if the family $\mathcal{F}$ is closed under addition and contains the constant 0 function then $\mathcal{M}_{add}(\mathcal{F}) = \mathcal{F}$, and similarly, if the family $\mathcal{F}$ is closed under multiplication and contains the constant 1 function, then $\mathcal{M}_{mult}(\mathcal{F}) = \mathcal{F}$. This is true also conversely.

If $\mathcal{F} = -\mathcal{F}$ (where $-\mathcal{F} = \{f; -f \in \mathcal{F}\}$) then $\mathcal{M}_{min}(\mathcal{F}) = -\mathcal{M}_{max}(\mathcal{F})$. Moreover, $\mathcal{M}_{latt}(\mathcal{F}) = \mathcal{M}_{max}(\mathcal{F}) \cap \mathcal{M}_{min}(\mathcal{F})$. Therefore, if we have knowledge of the maximal classes $\mathcal{M}_{max}(\mathcal{F})$ and $\mathcal{M}_{min}(\mathcal{F})$ then we have knowledge of $\mathcal{M}_{latt}(\mathcal{F})$, too. However, for some families (e.g. for quasicontinuous almost continuous (in the sense of Stallings) functions), we have knowledge of $\mathcal{M}_{latt}(\mathcal{F})$, however, a characterization of maximal classes with respect to maximum or minimum is an open problem.

Moreover, we can define maximal classes with respect to composition of functions: $\mathcal{M}_{out}(\mathcal{F}) = \{f; f \circ g \in \mathcal{F} \text{ for every } g \in \mathcal{F}\}$ and $\mathcal{M}_{in}(\mathcal{F}) = \{f; g \circ f \in \mathcal{F} \text{ for every } g \in \mathcal{F}\}$. Maximal classes with respect to composition were investigated e.g. in [3, 29, 37, 54, 55]. The main problem with characterizations of maximal classes is that the operator of maximal classes is not monotone. Hence, a characterizations of maximal classes can be unexpected and interesting. Unfortunately, the results concerning maximal classes are scattered throughout the literature. In this paper, we will deal with maximal classes for some families of functions that are generalizations of continuity.

Unless explicitly written, X and Y will be topological spaces. For a subset A of X, denote by $\mathrm{Cl}(A)$ the closure of A and by $\mathrm{Int}(A)$ the interior of A. In the results described below, unless explicitly written, we will assume that the functions are defined on $\mathbb{R}$.

2. Darboux and similar functions

Darboux functions

The oldest results on maximal classes are know for Darboux functions. Maximal additive and maximal multiplicative classes were characterized by T. Radaković in 1931. A function f defined on $\mathbb{R}$ is Darboux if whenever $a < b$ and c is any number between $f(a)$ and $f(b)$, there is a number $z \in (a,b)$ such that $f(z) = c$. Let $\mathcal{C}$ denote the family of all continuous functions, $\mathcal{D}$ the family of all Darboux functions and *Const* the family of all constant functions. Further, let usc denote upper semicontinuous functions and lsc lower semicontinuous functions.

Theorem 2.1 ([48]). *$\mathcal{M}_{add}(\mathcal{D}) = \mathcal{M}_{mult}(\mathcal{D}) = Const$.*

A characterization for maximum and minimum classes was given by J. Farková.

Theorem 2.2 ([11]). *$\mathcal{M}_{max}(\mathcal{D}) = \mathcal{D} \cap usc$, $\mathcal{M}_{min}(\mathcal{D}) = \mathcal{D} \cap lsc$ and $\mathcal{M}_{latt}(\mathcal{D}) = \mathcal{C}$.*

J. Jastrzębski characterized maximal additive families for some subclasses of Darboux functions. Denote the family of Darboux functions whose upper and lower boundary functions are continuous by $\mathcal{D}^C$, the family of functions which take on every real value in every interval by $\mathcal{D}^*$, and the family of functions which take on every real value $\mathfrak{c}$-times in every interval by $\mathcal{D}^{**}$. Further, let $Const_I$ be a family of functions such that there exixts a sequence of open intervals (I_k) such that $\bigcup_{k\in\mathbb{N}} I_k$ is dense in $\mathbb{R}$ and $f \restriction I_k$ is constant, and let $Const_{\mathfrak{c}I}$ be a family of functions such that there is a sequence (I_k) of open intervals and a sequence (A_k) of sets such that $\bigcup_{k\in\mathbb{N}} I_k$ is dense in $\mathbb{R}$, $A_k \subset I_k$, the cardinality of A_k is less that $\mathfrak{c}$ and $f \restriction (I_k \setminus A_k)$ is constant ([24]).

Theorem 2.3 ([24]). *$\mathcal{M}_{add}(\mathcal{D}^C) = \mathcal{C} \cap Const_I$, $\mathcal{M}_{add}(\mathcal{D}^*) = Const_I$ and $\mathcal{M}_{add}(\mathcal{D}^{**}) = Const_{\mathfrak{c}I}$.*

Darboux Baire one functions

Denote the family of all Baire one functions by $\mathcal{B}_1$. The maximal additive class for Darboux Baire one functions was characterized by A. M. Bruckner and J. Ceder, see also [8].

Theorem 2.4 ([9]). *$\mathcal{M}_{add}(\mathcal{D} \cap \mathcal{B}_1) = \mathcal{C}$.*

In [8], it is also shown that $\mathcal{C} \subsetneq \mathcal{M}_{mult}(\mathcal{D}\mathcal{B}_1)$. The problem of characterizing the maximal multiplicative family was solved by R. Fleissner.

Theorem 2.5 ([13]). *$\mathcal{M}_{mult}(\mathcal{D} \cap \mathcal{B}_1) = \mathcal{M}$.*

Here, $\mathcal{M}$ stands for the Fleissner family of all functions with the following property: if x_0 is a right-hand (left-hand) point of discontinuity of f, then $f(x_0) = 0$, and there exists a sequence (x_n) converging to x_0 such that $x_n > x_0$ $(x_n < x_0)$ and $f(x_n) = 0$.

Maximum and minimum classes were characterized again by J. Farková.

Theorem 2.6 ([11]). *$\mathcal{M}_{max}(\mathcal{D} \cap \mathcal{B}_1) = \mathcal{D} \cap usc$, $\mathcal{M}_{min}(\mathcal{D} \cap \mathcal{B}_1) = \mathcal{D} \cap lsc$ and $\mathcal{M}_{latt}(\mathcal{D}) = \mathcal{C}$.*

The characterization of maximal additive and multiplicative classes was extended for functions defined on the Euclidean space $\mathbb{R}^m$ by L. Mišík in [38] and later, for functions defined on some Banach spaces, in [39]. Let X be a topological space and let $\mathcal{B}$ be a base for the topology in X. A real function defined on X is called $\mathcal{B}$-Darboux if for each $A \in \mathcal{B}$, every $x, y \in \mathrm{Cl}(A)$ and each c between $f(x)$ and $f(y)$ there exists a point $z \in A$ such that $f(z) = c$. Denote the family of such functions by $\mathcal{D}_{\mathcal{B}}$ ([40]).

Theorem 2.7 ([39]). *Let X be a finite-dimensional strictly convex Banach space and let $\mathcal{B}$ be the system of all sets $a+U_r$, where $a \in X$, $U_r = \{x \in X; \|x\| < r\}$ and $r > 0$. Then, $\mathcal{M}_{add}(\mathcal{D}_{\mathcal{B}} \cap \mathcal{B}_1) = \mathcal{C}$ and $\mathcal{M}_{mult}(\mathcal{D}_{\mathcal{B}} \cap \mathcal{B}_1) = \mathcal{M}_M$, where $\mathcal{M}_M$ is the family of all $\mathcal{B}$-Darboux Baire one functions with the property: if $x \in X$ and $B \in \mathcal{B}$ are such that $x \in \mathrm{Cl}(B) \setminus B$ and if there is a sequence (C_n) of elements of $\mathcal{B}$ such that $x \in \mathrm{Cl}(C_n) \setminus C_n$, $C_{n+1} \subset C_n \subset B$, $\lim\limits_{n\to\infty} \mathrm{diam}(C_n) = 0$ and $\sup\limits_n \inf f(C_n) < \inf\limits_n \sup f(C_n)$, then $f(x) = 0$, and there exists a sequence (x_k) of points of B such that $f(x_k) = 0$ for all k.*

Here, $\mathrm{diam}(C)$ is the diameter of the set C.

Connectivity and functionally connected functions

A function $f : X \to \mathbb{R}$ is connectivity if the graph of f restricted to C is a connected subset of $X \times \mathbb{R}$ for each connected subset C of $\mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is functionally connected if for each $a < b$ and each continuous function $g : [a, b] \to \mathbb{R}$ with $(f(a) - g(a))(f(b) - g(b)) < 0$, there is a point $c \in [a, b]$ with $f(c) = g(c)$ ([25]). Denote by Con the family of all connected functions and by $Fcon$ the family of all functionally connected functions. Characterizations of maximal classes for these families are similar, however, a characterization of maximal classes for maximum and minimum seems to be an open problem.

Theorem 2.8 ([25]). *$\mathcal{M}_{add}(Con) = \mathcal{M}_{add}(Fcon) = \mathcal{C}$.*

Theorem 2.9 ([26]). *$\mathcal{M}_{latt}(Con) = \mathcal{M}_{latt}(Fcon) = \mathcal{C}$, $\mathcal{M}_{mult}(Con) = \mathcal{M}_{mult}(Fcon) = \mathcal{M}$, $\mathcal{M}_{max}(Fcon) = \mathcal{D} \cap usc$ and $\mathcal{M}_{min}(Fcon) = \mathcal{D} \cap lsc$.*

Extendable functions

A function f is extendable if there exists a connectivity function $F : \mathbb{R} \times [0, 1] \to \mathbb{R}$ such that $F(x, 0) = f(x)$ for every $x \in \mathbb{R}$. Denote by Ext the family of all extendable functions.

Theorem 2.10 ([27]). *$\mathcal{M}_{add}(Ext) = \mathcal{M}_{latt}(Ext) = \mathcal{C}$ and $\mathcal{M}_{mult}(Ext) = \mathcal{M}$.*

Functions with perfect road

A function $f : \mathbb{R} \to \mathbb{R}$ has a perfect road if for every $x \in \mathbb{R}$ there exists a perfect set P having x to be a bilateral limit such that $f \restriction P$ is continuous at x ([35]). Denote functions with perfect road by PR. Maximal additive and multiplicative classes were characterized by K. Banaszewski.

Theorem 2.11 ([2]). *$\mathcal{M}_{add}(PR) = \mathcal{C}$ and $\mathcal{M}_{mult}(PR) = \mathcal{M}$.*

Young (peripherally continuous) functions

A function $f : \mathbb{R} \to \mathbb{R}$ is a Young function (peripherally continuous function) if for each $x \in \mathbb{R}$ there exist sequences (x_n) and (y_n) such that $x_n < x$, $y_n > x$, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x$ and $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n) = f(x)$. Denote the family of all Young functions by PC. Maximal classes were characterized by K. Banaszewski.

Theorem 2.12 ([2]). *$\mathcal{M}_{add}(PC) = \mathcal{M}_{max}(PC) = \mathcal{M}_{min}(PC) = \mathcal{M}_{latt}(PC) = \mathcal{C}$ and $\mathcal{M}_{mult}(PC) = \mathcal{M}$.*

CIVP functions and SCIVP functions

A function $f : \mathbb{R} \to \mathbb{R}$ has the Cantor intermediate value property if for every $x, y \in \mathbb{R}$ and for each Cantor set K between $f(x)$ and $f(y)$ there exists a Cantor set C between x and y such that $f(C) \subset K$ ([15]). A function f has the strong Cantor intermediate value property if for every $x, y \in \mathbb{R}$ and for each Cantor set K between $f(x)$ and $f(y)$, there exists a Cantor set C between x and y such that $f(C) \subset K$ and the restriction $f \restriction C$ is continuous. Denote the family of functions with the Cantor intermediate value property by CIVP and with the strong Cantor intermediate value property by SCIVP.

Theorem 2.13 ([4]). *Assume CH. Then, $\mathcal{M}_{add}(CIVP) = \mathcal{M}_{mult}(CIVP) = \mathcal{M}_{latt}(CIVP) = \mathcal{M}_{add}(\mathcal{D} \cap CIVP) = \mathcal{M}_{mult}(\mathcal{D} \cap CIVP) = \mathcal{M}_{latt}(\mathcal{D} \cap CIVP) = Const$.*

Theorem 2.14 ([14]). *Assume CH. Then, $\mathcal{M}_{add}(SCIVP) = \mathcal{M}_{mult}(SCIVP) = Const$.*

Almost continuous (Stallings) functions

A function f is almost continuous (in the sense of Stallings) if for every open set $G \subset \mathbb{R} \times \mathbb{R}$ containing the graph of f, there is a continuous function g such that the graph of g lies in G ([52]). Denote the family of all almost continuous functions (in the sense of Stallings) by ACS. For these functions we known all maximal classes.

Theorem 2.15 ([43]). *$\mathcal{M}_{add}(ACS) = \mathcal{C}$.*

Theorem 2.16 ([26]). *$\mathcal{M}_{mult}(ACS) = \mathcal{M}_{latt}(ACS) = \mathcal{C}$.*

The maximal classes for maximum and minimum were characterized only recently (it is an affirmative answer to a conjecture in [26]).

Theorem 2.17 ([34]). *$\mathcal{M}_{max}(ACS) = \mathcal{D} \cap usc$ and $\mathcal{M}_{min}(ACS) = \mathcal{D} \cap lsc$.*

3. Quasicontinuous and similar functions

Quasicontinuous functions

Whereas the great part of results on maximal classes for Darboux-like functions concerns functions defined on $\mathbb{R}$, the great part of results on maximal classes for quasicontinuous-like functions concerns functions defined on topological spaces. Let X be a topological space and let $C(f)$ denote the set of all continuity points of f.

Recall that a function $f : X \to \mathbb{R}$ is said to be quasicontinuous at a point x if for each neighbourhood U of x and each $\varepsilon > 0$, there is an open nonempty set $G \subset U$ such that $f(G) \subset (f(x)-\varepsilon, f(x)+\varepsilon)$ ([28]). A function f is quasicontinuous if it is such at every point. Denote the family of all quasicontinuous functions by $\mathcal{Q}$ and the family of bounded functions by b. Further, denote by $Q(f)$ the set of all quasicontinuity points of f.

The maximal additive class and the maximal lattice-like class for quasicontinuous functions was decribed by Z. Grande and L. Sołtysik in [22].

Theorem 3.1 ([22]). *Let X be a topological space. Then $\mathcal{M}_{add}(\mathcal{Q}) = \mathcal{M}_{latt}(\mathcal{Q}) = \mathcal{C}$.*

The maximal multiplicative family for a complete metric space X was characterized by Z. Grande and L. Sołtysik in [22] and later, by Z. Grande, for an arbitrary topological space X. Let $\mathcal{Q}_M$ stand for the family of all quasicontinuous functions $f : X \to \mathbb{R}$ with the property: if f is discontinuous at x, then $f(x) = 0$ and $x \in \mathrm{Cl}(C(f) \cap f^{-1}(0))$. Further, let $\mathcal{Q}_{bM}$ denote the family of all quasicontinuous functions with the property: if f is discontinuous at x then $f(x) = 0$.

Theorem 3.2 ([21]). *Let X be a topological space. Then, $\mathcal{M}_{mult}(\mathcal{Q}) = \mathcal{Q}_M$ and $\mathcal{M}_{mult}(b \cap \mathcal{Q}) = \mathcal{Q}_{bM}$.*

Denote the family of all functions for which the set $X \setminus C(f)$ is nowhere dense by $\mathcal{C}^*$, and the family of all functions for which the set $X \setminus Q(f)$ is nowhere dense by $\mathcal{Q}^*$. Whereas $\mathcal{M}_{add}(\mathcal{Q}) \neq \mathcal{M}_{mult}(\mathcal{Q})$, for the family $\mathcal{Q}^*$, these maximal classes coincide.

Theorem 3.3 ([7]). *Let X be a Baire space. Then, $\mathcal{M}_{add}(\mathcal{Q}^*) = \mathcal{M}_{mult}(\mathcal{Q}^*) = \mathcal{C}^*$.*

Maximal classes for maximum and minimum were described by T. Natkaniec.

Theorem 3.4 ([42]). *Let X be a topological space. Then, $\mathcal{M}_{max}(\mathcal{Q}) = \mathcal{M}_{min}(\mathcal{Q}) = \mathcal{C}$.*

Upper and lower quasicontinuous functions

A function $f : X \to \mathbb{R}$ is upper (lower) quasicontinuous at $x \in X$ if for every positive $\varepsilon > 0$ and every neighbourhood U of x there exists a nonempty open set $G \subset U$ such that $f(y) < f(x) + \varepsilon$ $(f(y) > f(x) - \varepsilon)$ for each $y \in G$ ([10]). Let $\mathcal{Q}_E$ denote the family of all functions which are both upper and lower quasicontinuous at each $x \in X$. Notice that $\mathcal{Q}_E$ is a nowhere dense set in $\mathcal{Q}$ in the topology of the uniform convergence (for $X = \mathbb{R}$). Maximal additive and lattice-like classes were characterized by E. Strońska in [53]. In her paper, we can find some necessary and some sufficient conditions for the maximal multiplicative class, however, its characterization is still open.

Theorem 3.5 ([53]). *Let X be a topological space. Then, $\mathcal{M}_{add}(\mathcal{Q}_E) = \mathcal{M}_{max}(\mathcal{Q}_E) = \mathcal{M}_{min}(\mathcal{Q}_E) = \mathcal{M}_{latt}(\mathcal{Q}_E) = \mathcal{C}$.*

Symmetrically quasicontinuous functions

A function $f : X \times Y \to \mathbb{R}$ (X and Y are topological spaces) is said to be quasicontinuous at (x, y) with respect to first (second) coordinate if for every neighbourhoods U, V and W of x, y and $f(x, y)$, respectively, there are nonempty open sets G and H such that $x \in G \subset U$, $H \subset V$ ($G \subset U$, $y \in H \subset V$) and $f(G \times H) \subset W$. A function f is symmetrically quasicontinuous at (x, y) if it is quasicontinuous both with respect to the first and the second coordinate ([46]). Denote by $\mathcal{Q}_{sx}$, $\mathcal{Q}_{sy}$ and $\mathcal{Q}_{ss}$ the family of all functions which are quasicontinuous with respect to first coordinate, quasicontinuous with respect to second coordinate, symmetrically quasicontinuous at each point, respectively. Further, let $\mathcal{Q}_{ss0}$ denote the family of all functions from $\mathcal{Q}_{ss}$ such that $f(x, y) \neq 0$ for each $(x, y) \in X \times Y$. For $x \in X$, a function $f_x : Y \to \mathbb{R}$, $f_x(y) = f(x, y)$ is the x-section of f; similarly, the y-section $f^y : X \to \mathbb{R}$ is defined as $f^y(x) = f(x, y)$. Maximal additive classes for $\mathcal{Q}_{sx}$ and $\mathcal{Q}_{sy}$ are characterized for arbitrary topological spaces X and Y.

Theorem 3.6 ([19]). *Let X and Y be topological spaces. Then,*

$$\mathcal{M}_{add}(\mathcal{Q}_{sx}) = \{g \in \mathcal{Q}_{sx}; \textit{sections } g_x \textit{ are continuous}\}.$$

Similarly,

$$\mathcal{M}_{add}(\mathcal{Q}_{sy}) = \{g \in \mathcal{Q}_{sy}; \textit{sections } g^y \textit{ are continuous}\}.$$

The investigation of the maximal additive class for $\mathcal{Q}_{ss}$ is more complicated. Let $(x, y) \in X \times Y$ be a point. We say that a closed set $A \subset X \times Y$ belongs to the family $S(x, y)$ $[(P(x, y)]$ if we have:
$A_x = \{y\}$ $\quad [A^y = \{x\}]$,
$x \in \mathrm{Cl}((\mathrm{Int}(A))^y)$ $\quad [y \in \mathrm{Cl}((\mathrm{Int}(A))_x)\ ]$,
and for each point $(u, v) \in A \backslash \{(x, y)\}$, we have $u \in \mathrm{Cl}(\mathrm{Int}(A))^v)$ and $v \in \mathrm{Cl}(\mathrm{Int}(A))_u)$, where $A_x = \{t \in Y; (x, t) \in A\}$ and $A^y = \{t \in X; (t, y) \in A\}$.

Theorem 3.7 ([19]). *Let X and Y be topological spaces such that, for each point $(x, y) \in X \times Y$, the families $S(x, y)$ and $P(x, y)$ are nonempty. Then,*

$$\mathcal{M}_{add}(\mathcal{Q}_{ss}) = \{f \in \mathcal{Q}_{ss}; f \textit{ is separately continuous}\}.$$

There is an open problem whether above the theorem holds for arbitrary topological spaces. The Euclidean plane ($X = Y = \mathbb{R}$) satisfies the assumptions of this theorem. Moreover, in his paper, we can find some necessary and some sufficient conditions for the maximal multiplicative classes for families $\mathcal{Q}_{sx}$, $\mathcal{Q}_{sy}$ and $\mathcal{Q}_{ss}$, however, their characterization is open. For the family $\mathcal{Q}_{ss0}$ we have a characterization.

Theorem 3.8 ([19]). *Let X and Y be topological spaces such that for each point $(x, y) \in X \times Y$ the families $S(x, y)$ and $P(x, y)$ are nonempty. Then,*

$$\mathcal{M}_{mult}(\mathcal{Q}_{ss0}) = \{f \in \mathcal{Q}_{ss0}; f \textit{ is separately continuous}\}.$$

Strongly quasicontinuous functions

For a measurable set $E \subset \mathbb{R}$ let $\ell(E)$ stand the Lebesgue measure of E. For a measurable set E and $x \in \mathbb{R}$, the numbers

$$d_l(E,x) = \liminf_{t\to 0^+, k\to 0^+} \frac{\ell(E\cap [x-t,x+k])}{k+t},$$

$$d^u(E,x) = \limsup_{t\to 0^+, k\to 0^+} \frac{\ell(E\cap [x-t,x+k])}{k+t}$$

are called the upper and lower density of E at x, respectively. If $d_l(E,x) = d^u(E,x)$, we call this number the density of E at x and denote it by $d(E,x)$. If $d(E,x) = 1$, we say that x is a density point of E. The family

$$\mathcal{T}_d = \{A \subset \mathbb{R}; A \text{ is measurable and every point } x \in A \text{ is a density point of } A\}$$

is a topology called the density topology. A function $f : \mathbb{R} \to \mathbb{R}$ is called approximately continuous at $x \in \mathbb{R}$ if there is a measuarble set E containing x such that $d(E,x) = 1$ and the restriction $f \restriction E$ is continuous at x. Let $D_{ap}(f)$ be the set of all points at which f is not approximately continuous. A function f is approximately continuous if $D_{ap}(f) = \emptyset$. Equivalently, a function f is approximately continuous if it is continuous as the application from $\mathbb{R}$ equipped with the density topology $\mathcal{T}_d$. Of course, approximately continuous functions are closed under addition and multiplication and so maximal classes are obvious. Let $\mathcal{A}$ denote the family of all approximately continuous functions and $\mathcal{C}_{ae}$ the family of functions continuous almost everwhere (i.e. such that $\ell(D(f) = 0)$.

A function $f : \mathbb{R} \to \mathbb{R}$ is strongly quasicontinuous if for every $x \in \mathbb{R}$, for every set $A \in \mathcal{T}_d$ containing x and for every $\varepsilon > 0$ there is an open interval I such that $I \cap A \neq \emptyset$ and $|f(t) - f(x)| < \varepsilon$ for all $t \in A \cap I$. A function $f : \mathbb{R} \to \mathbb{R}$ is s_1-strongly quasicontinuous (s_2-strongly quasicontinuous) if for every $x \in \mathbb{R}$, for every set $A \in \mathcal{T}_d$ containing x and for every $\varepsilon > 0$ there exists an open interval I such that $I \cap A \neq \emptyset$, $I \cap A \subset C(f)$ ($I \cap A \subset \mathbb{R} \setminus D_{ap}(f)$) and $|f(t) - f(x)| < \varepsilon$ for all $t \in A \cap I$. Denote the families of strongly quasicontinuous functions, s_1-strongly quasicontinuous functions and s_2-strongly quasicontinuous functions by $\mathcal{Q}_s$, $\mathcal{Q}_{s_1}$ and $\mathcal{Q}_{s_2}$, respectively ([18]). Maximal families were investigated by E. Strońska.

Theorem 3.9 ([55]). *$\mathcal{M}_{add}(\mathcal{Q}_s) = \mathcal{M}_{max}(\mathcal{Q}_s) = \mathcal{M}_{min}(\mathcal{Q}_s) = \mathcal{M}_{latt}(\mathcal{Q}_s) = \mathcal{Q}_s \cap \mathcal{A} \cap \mathcal{C}_{ae}$, $\mathcal{M}_{add}(\mathcal{Q}_{s_1}) = \mathcal{M}_{max}(\mathcal{Q}_{s_1}) = \mathcal{M}_{min}(\mathcal{Q}_{s_1}) = \mathcal{M}_{latt}(\mathcal{Q}_{s_1}) = \mathcal{Q}_{s_1} \cap \mathcal{A} \cap \mathcal{C}_{ae}$ and $\mathcal{M}_{add}(\mathcal{Q}_{s_2}) = \mathcal{M}_{max}(\mathcal{Q}_{s_2}) = \mathcal{M}_{min}(\mathcal{Q}_{s_2}) = \mathcal{M}_{latt}(\mathcal{Q}_{s_2}) = \mathcal{Q}_{s_2} \cap \mathcal{A} \cap \mathcal{C}_{ae}$.*

Let $\mathcal{T}_{ae}$ be the family of all sets $A \in \mathcal{T}_d$ such that $\ell(A \setminus \mathrm{Int}(A)) = 0$. The family $\mathcal{T}_{ae}$ is also topology. Let $\mathcal{M}_Q$ denote the family of all functions with this property: if f is not $\mathcal{T}_{ae}$-continuous at $x \in \mathbb{R}$ (where f is considered as the application from $\mathbb{R}$ equipped with the topology $\mathcal{T}_{ae}$) then $f(x) = 0$ and $d^u(\{t \in \mathbb{R}; f(t) = 0\}, x) > 0$.

Theorem 3.10 ([55]). *$\mathcal{M}_{mult}(\mathcal{Q}_s) = \mathcal{Q}_s \cap \mathcal{M}_Q$, $\mathcal{M}_{mult}(\mathcal{Q}_{s_1}) = \mathcal{Q}_{s_1} \cap \mathcal{M}_Q$ and $\mathcal{M}_{mult}(\mathcal{Q}_{s_2}) = \mathcal{Q}_{s_2} \cap \mathcal{M}_Q$.*

The results were later extended for functions $f : \mathbb{R}^m \to \mathbb{R}$ by E. Strońska in [54].

4. Darboux quasicontinuous and similar functions

Darboux quasicontinuous functions

Maximal classes for functions both Darboux and quasicontinuous were characterized by T. Natkaniec.

Theorem 4.1 ([41]). *$\mathcal{M}_{add}(\mathcal{D} \cap \mathcal{Q}) = \mathcal{M}_{mult}(\mathcal{D} \cap \mathcal{Q}) = Const$, $\mathcal{M}_{max}(\mathcal{D} \cap \mathcal{Q}) = \mathcal{D} \cap usc$, $\mathcal{M}_{min}(\mathcal{D} \cap \mathcal{Q}) = \mathcal{D} \cap lsc$ and $\mathcal{M}_{latt}(\mathcal{D} \cap \mathcal{Q}) = \mathcal{C}$.*

Darboux cliquish functions

A function $f : X \to \mathbb{R}$ is said to be cliquish if for each point x and for each neighbourhood U of x and each $\varepsilon > 0$ there is an open nonempty set $G \subset U$ such that $|f(y) - f(z)| < \varepsilon$ for all $y, z \in G$. A function f defined on $\mathbb{R}$ is cliquish if the set of continuity points of f is dense in $\mathbb{R}$. Denote the family of cliquish functions by $Cliq$. The family $Cliq$ is closed under addition and multiplication, so maximal families are obvious. Further, let $\mathcal{B}_\alpha$ denote the family of all functions in Baire class α and $\mathcal{L}$ the family of all measurable functions. Maximal classes for Darboux cliquish functions were investigated by A. Maliszewski. A function f is in honorary Baire class two if there is a function g in Baire class one which equals f for all but countably many arguments. Denote the family of all honorary Baire class two functions by $\mathcal{B}_2^h$. The maximal additive class for Darboux honorary Baire class two functions was characterized by I. Pokorný in [47], the maximal multiplicative class by A. Maliszewski in [33].

Theorem 4.2 ([47]). *$\mathcal{M}_{add}(\mathcal{B}_2^h) = Const$.*

Theorem 4.3 ([33]). *Let $\alpha \geqslant 2$. Then, $\mathcal{M}_{add}(\mathcal{D} \cap Cliq) = \mathcal{M}_{add}(\mathcal{L} \cap \mathcal{D} \cap Cliq) = \mathcal{M}_{add}(\mathcal{D} \cap Cliq \cap \mathcal{B}_\alpha) = \mathcal{M}_{add}(\mathcal{D} \cap \mathcal{B}_\alpha) = Const$.*

Theorem 4.4 ([33]). *Let $\alpha \geqslant 2$. Then, $\mathcal{M}_{mult}(\mathcal{D} \cap Cliq) = \mathcal{M}_{mult}(\mathcal{L} \cap \mathcal{D} \cap Cliq) = \mathcal{M}_{mult}(\mathcal{D} \cap Cliq \cap \mathcal{B}_\alpha) = \mathcal{M}_{mult}(\mathcal{D} \cap \mathcal{B}_\alpha) = \mathcal{M}_{mult}(\mathcal{D} \cap \mathcal{B}_2^h) = Const$.*

Darboux almost continuous (Stallings) functions

T. Natkaniec characterized maximal additive, multiplicative and lattice-like classes for these families. A problem to characterize the maximal classes with respect to maximum (minimum) remains open.

Theorem 4.5 ([41]). *$\mathcal{M}_{add}(ACS \cap \mathcal{Q}) = \mathcal{M}_{latt}(ACS \cap \mathcal{Q}) = \mathcal{C}$ and $\mathcal{M}_{mult}(ACS \cap \mathcal{Q}) = \mathcal{M}$.*

Connectivity quasicontinuous functions

Similarly, maximal additive, multiplicative and lattice-like families were characterized by T. Natkaniec. And, similarly, a problem of characterizing the maximal maximum (minimum) class is open.

Theorem 4.6 ([41]). *$\mathcal{M}_{add}(Con \cap \mathcal{Q}) = \mathcal{M}_{latt}(Con \cap \mathcal{Q}) = \mathcal{C}$ and $\mathcal{M}_{mult}(Con \cap \mathcal{Q}) = \mathcal{M}$.*

Strong Świątkowski and extra strong Świątkowski functions

A function $f : \mathbb{R} \to \mathbb{R}$ is a strong Świątkowski function if whenever $a < b$ and c is between $f(a)$ and $f(b)$, there is $x \in (a, b) \cap C(f)$. A function f is an extra strong Świątkowski function if $f([a, b]) = f([a, b] \cap C(f))$ for all $a < b$. Let $\mathcal{S}_s$ denote the family of all strong Świątkowski functions and $\mathcal{S}_{es}$ the family of all extra strong Świątkowski functions. Evidently, $\mathcal{S}_{es} \subset \mathcal{S}_s \subset \mathcal{D} \cap \mathcal{Q}$. Maximal families were characterized by P. Szczuka.

Theorem 4.7 ([56]). *$\mathcal{M}_{add}(\mathcal{S}_s) = \mathcal{M}_{add}(\mathcal{S}_{es}) = \mathcal{M}_{mult}(\mathcal{S}_s) = \mathcal{M}_{mult}(\mathcal{S}_{es}) = \mathcal{M}_{max}(\mathcal{S}_s) = \mathcal{M}_{max}(\mathcal{S}_{es}) = \mathcal{M}_{min}(\mathcal{S}_s) = \mathcal{M}_{min}(\mathcal{S}_{es}) = \mathcal{M}_{latt}(\mathcal{S}_s) = \mathcal{M}_{latt}(\mathcal{S}_{es}) = Const$.*

5. Other generalizations of continuity

Functions with closed graph

A function $f : X \to \mathbb{R}$ has closed graph if the graph of f is a closed subset of $X \times \mathbb{R}$. Let $\mathcal{U}$ denote the family of functions with closed graph. Maximal additive and multiplicative families were characterized by R. Menkyna.

Theorem 5.1 ([36]). *Let X be a topological space. Then, $\mathcal{M}_{add}(\mathcal{U}) = \mathcal{C}$.*

Theorem 5.2 ([36]). *Let X be a locally compact normal topological space. Then, $\mathcal{M}_{mult}(\mathcal{U}) = \{f \in \mathcal{C}; f^{-1}(0)$ is an open set $\}$.*

Quasicontinuous functions with closed graph

Maximal families for these families were characterized by W. Sieg.

Theorem 5.3 ([51]). *Let $X = \mathbb{R}$. Then, $\mathcal{M}_{add}(\mathcal{Q} \cap \mathcal{U}) = \mathcal{C}$, $\mathcal{M}_{mult}(\mathcal{Q} \cap \mathcal{U}) = \{f \in \mathcal{C}; f^{-1}(0)$ is an open set $\}$, and $\mathcal{M}_{max}(\mathcal{Q} \cap \mathcal{U}) = \mathcal{M}_{min}(\mathcal{Q} \cap \mathcal{U}) = \mathcal{M}_{latt}(\mathcal{Q} \cap \mathcal{U}) = \emptyset$.*

It remains open to question whether this theorem holds for an arbitrary topological space.

Graph continuous

A function $f : X \to \mathbb{R}$ is graph continuous if there exists a continuous function $g : X \to \mathbb{R}$ such that the closure of the graph of f contains the graph of f. Let Gr be the family of graph continuous functions. Maximal families were characterized by K. Sakálová.

Theorem 5.4 ([50]). *Let X be a connected Hausdorff topological space. Then, $\mathcal{M}_{add}(Gr) = Gr \cap Cliq$.*

Simply continuous functions

A function $f : X \to \mathbb{R}$ is simply continuous if for each open set V in $\mathbb{R}$, the set $f^{-1}(V)$ is the union of an open set and a nowhere dense set in X ([5]). Let $\mathcal{S}$ denote the family of all simply continuous functions. We have $\mathcal{Q} \subset \mathcal{S}$ and, if X is Baire, $\mathcal{S} \subset Cliq$. Further, let $Const_G$ be the family of all functions for which the set $\bigcup \mathcal{G}(f)$ is dense in X, where $\mathcal{G}(f) = \{G \subset X; G \text{ is open and } f \text{ is constant on } G\}$. (Therefore, for $X = \mathbb{R}$ we have the Jastrzębski family $Const_I$.)

Theorem 5.5 ([7]). *Let X be a Baire space such that the family of all connected open sets is a π-base for X and there is a dense set of first category in X. Then, $\mathcal{M}_{add}(\mathcal{S}) = \mathcal{M}_{mult}(\mathcal{S}) = Const_G$.*

This theorem does not hold for an arbitrary topological space X. For maxima and minima we have

Theorem 5.6 ([6]). *Assume that X is a topological space with the following property:*

(*) if (X_n) is a partition of X such that $\bigcup_{n\in M} X_n$ is simply open for each $M \subset \mathbb{N}$, and G is a nonempty open set in X, then $G \cap \operatorname{Int} X_n \neq \emptyset$ for some $n \in \mathbb{N}$.

Then $\mathcal{M}_{max}(\mathcal{S}) = \mathcal{M}_{min}(\mathcal{S}) = \mathcal{M}_{latt}(\mathcal{S}) = \mathcal{S}$.

If X is either a Baire space, or has a locally countable π-base, then it possesses the property (*), however, there are topological spaces which are neither Baire nor have countable π-base, but still possess the property (*). Moreover, there are topological spaces which do not satisfy condition (*). It is an open problem whether this theorem is true for an arbitrary topological space.

Almost continuous functions (Husain)

A function $f : X \to \mathbb{R}$ is precontinuous or almost continuous (in the sense of Husain) if for each point $x \in X$ and for every $\varepsilon > 0$ we have $x \in \operatorname{Int}(\operatorname{Cl}(f^{-1}(f(x) - \varepsilon, f(x) + \varepsilon)))$. Let ACH denote the family of all functions almost continuous in the sense of Husain. The maximal families were characterized by Z. Grande.

Theorem 5.7 ([16]). *Let X be a topological space. Then, $\mathcal{M}_{add}(ACH) = \mathcal{M}_{mult}(ACH) = \mathcal{M}_{max}(ACH) = \mathcal{M}_{min}(ACH) = \mathcal{M}_{latt}(ACH) = \mathcal{C}$.*

ρ-upper continuous and similar functions

A function $f : \mathbb{R} \to \mathbb{R}$ is 1-upper continuous if for every $x \in \mathbb{R}$ there exists a measurable set E containing x such that $d^u(E, x) = 1$ and the restriction $f \restriction E$ is continuous at x. For $0 < \rho < 1$, a function f is ρ-upper continuous if there is a measurable set E containing x such that $d(E, x) > \rho$ and the restriction $f \restriction E$ is continuous at x. Let $\mathcal{C}_1$ denote the family of all 1-upper continuous functions and $\mathcal{C}_\rho$ denote the family of all ρ-upper continuous functions. Maximal classes for families $\mathcal{C}_1$ and $\mathcal{C}_\rho$ were investigated by S. Kowalczyk and K. Nowakowska.

Theorem 5.8 ([31])**.** *Let $0 < \rho < 1$. Then $\mathcal{M}_{add}(\mathcal{C}_\rho) = \mathcal{A}$.*

The family $\mathcal{A}$ is only a proper subset of $\mathcal{M}_{add}(\mathcal{C}_1)$. A measurable set E is called sparse at $x \in \mathbb{R}$ if for every measurable set $F \subset \mathbb{R}$, if $d^u(F, x) < 1$, then $d^u(E \cup F, x) < 1$ ([12]). A function $f : \mathbb{R} \to \mathbb{R}$ is T^*-continuous if for each $x \in \mathbb{R}$ and for each $\varepsilon > 0$, the complement of the set $\{y \in \mathbb{R}; |f(y) - f(x)| < \varepsilon\}$ is sparse at x. Let $\mathcal{C}_{T^*}$ denote the family of all T^*-continuous functions.

Theorem 5.9 ([31])**.** *$\mathcal{M}_{add}(\mathcal{C}_1) = \mathcal{C}_{T^*}$.*

For $0 < \rho < 1$, let $\mathcal{Z}_\rho$ be the family of all measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that for each $x \in D_{ap}(f)$ we have $f(x) = 0$, and, for each measurable set E such that $f^{-1}(0) \subset E$ and $d^u(E, x) > \rho$, we have $\lim\limits_{\varepsilon \to 0^+} d^u(E \cap \{y \in \mathbb{R}; |f(y)| < \varepsilon\}, x) > \rho$. Let $\mathcal{Z}_1$ be the family of all measurable functions such that for each point x at which f is not T^*-continuous we have $f(x) = 0$ and, for each measurable set E such that $f^{-1}(0) \subset E$ and $d^u(E, x) = 1$ and for each $\varepsilon > 0$ we have $d^u(E \cap \{y \in \mathbb{R}; |f(y)| < \varepsilon\}, x) = 1$. The family $\mathcal{A}$ is a proper subset of $\mathcal{Z}_\rho$ and the family $\mathcal{C}_{T^*}$ is a proper subset of $\mathcal{Z}_1$.

Theorem 5.10 ([31])**.** *$\mathcal{M}_{mult}(\mathcal{C}_1) = \mathcal{Z}_1$ and $\mathcal{M}_{mult}(\mathcal{C}_\rho) = \mathcal{Z}_\rho$.*

Similar functions are defined in [45]. Let $0 < \lambda \leqslant \rho < 1$. A function $f : \mathbb{R} \to \mathbb{R}$ is called $[\lambda, \rho]$-continuous if for each $x \in \mathbb{R}$ there exists a measurable set E containing x such that $d_l(E, x) > \lambda$, $d^u(E, x) > \rho$ and $f \restriction E$ is continuous at x. Let $\mathcal{C}_{[\lambda,\rho]}$ be the family of all $[\lambda, \rho]$-continuous functions. Maximal additive families for these functions were characterized in [32].

Theorem 5.11 ([32])**.** *$\mathcal{M}_{add}(\mathcal{C}_{[\lambda,\rho]}) = \mathcal{A}$.*

For $0 < \lambda \leqslant \rho < 1$, let $\mathcal{Z}_{[\lambda,\rho]}$ be the family of all measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that for each $x \in D_{ap}(f)$ we have $f(x) = 0$, and for each measurable set E such that $f^{-1}(0) \subset E$, $d_l(E, x) > \lambda$ and $d^u(E, x) > \rho$ we have $\lim\limits_{\varepsilon \to 0^+} d_l(E \cap \{y \in \mathbb{R}; |f(y) - f(x)| < \varepsilon\}, x) > \lambda$ and $\lim\limits_{\varepsilon \to 0^+} d^u(E \cap \{y \in \mathbb{R}; |f(y) - f(x)| < \varepsilon\}, x) > \rho$. The family $\mathcal{A}$ is a proper subset of $\mathcal{Z}_{[\lambda,\rho]}$.

Theorem 5.12 ([32])**.** *$\mathcal{M}_{mult}(\mathcal{C}_{[\lambda,\rho]}) = \mathcal{Z}_{[\lambda,\rho]}$.*

Theorem 5.13 ([32])**.** *$\mathcal{M}_{max}(\mathcal{C}_{[\lambda,\rho]}) = \mathcal{M}_{min}(\mathcal{C}_{[\lambda,\rho]}) = \mathcal{M}_{latt}(\mathcal{C}_{[\lambda,\rho]}) = \mathcal{A}$.*

A function $f : \mathbb{R} \to \mathbb{R}$ is called $[0]$-continuous if for each $x \in \mathbb{R}$ there exists a measurable set E containing x such that $d_l(E, x) > 0$ and $f \restriction E$ is continuous at x. Let $\mathcal{C}_{[0]}$ be the family of all $[0]$-continuous functions. For a function f, let $T(f)$ be the set of all $x \in \mathbb{R}$ such that for each measurable set E with $d_l(E, x) > 0$

we have $\lim_{\varepsilon\to 0^+} d_l(E \cap \{y \in \mathbb{R}; |f(y) - f(x)| < \varepsilon\}, x) > 0$ ([30]). Let $\mathcal{T}_{[0]}$ denote the family of all functions for which $T(f) = \mathbb{R}$. Further, let $\mathcal{W}_{[0]}$ be the family of all measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that for each $x \notin T(f)$, we have $f(x) = 0$, and for each measurable set E such that $f^{-1}(0) \subset E$ and $d_l(E, x) > 0$, we have $\lim_{\varepsilon\to 0^+} d_l(E\cap\{y \in \mathbb{R}; |f(y)-f(x)| < \varepsilon\}, x) > 0$. We have $\mathcal{A} \subsetneq \mathcal{T}_{[0]} \subsetneq \mathcal{C}_{T^*}$ and $\mathcal{W}_{[0]} \subsetneq \mathcal{T}_{[0]}$.

Theorem 5.14 ([30]). *$\mathcal{M}_{add}(\mathcal{C}_{[0]}) = \mathcal{T}_{[0]}$ and $\mathcal{M}_{mult}(\mathcal{C}_{[0]}) = \mathcal{W}_{[0]}$.*

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